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The Homogeneous Gravitational Force Field in General Relativity

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Abstract

The concept of a gravitational force in General Relativity is reintroduced. The theory of space-like congruences is established and is applied to the discussion of the existence of a homogeneous gravitational force-field in curved space-time. It is shown in vacuum (vanishing Ricci tensor) that such a force-field cannot exist.

1. *Introduction*

A fundamental concept of Newtonian theory is the *homogeneous* gravitational force-field. In this paper we want to consider how much of this concept can be brought over into the General Theory of Relativity. However, a central idea of General Relativity is to eliminate gravitation by treating its effects as results of the space-time curvature. Consequently the force-field itself has first to be reconstituted out of space-time concepts. It can be reintroduced by considering the effects of space-time curvature on a freely moving test particle, or one attached by a spring, as measured by an observer. The history of the observer will be thereby described by a time-like world line and the gravitational 3-force on the test particle will be a space-like vector in the orthogonal 3-rest space of the observer. Essential for the mathematical treatment of the gravitational force-field is therefore the theory of time-like and space-like congruences.

2. *Time-like Congruences*

The world lines of observers form a time-like congruence with a normalised tangent vector field u^α ($u^\alpha u_\alpha = +1$). Rigid rotation, shear and expansion of this congruence are given by (cf. Ehlers & Kundt, 1962; Ellis, 1971)†

$$\omega_{\alpha\beta} = u_{[\alpha;\beta]} - \dot{u}_{[\alpha} u_{\beta]}$$

† Signature $(- - - +)$, $\alpha, \beta, \ldots = 1, 2, 3, 4$. $A_{(\alpha\beta)} = \frac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha})$, $A_{[\alpha\beta]} = \frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha})$, and a semicolon denotes covariant differentiation.

$$\sigma_{\alpha\beta} = u_{(\alpha;\beta)} - \dot{u}_{(\alpha} u_{\beta)} - \tfrac{1}{3} u^{\gamma}{}_{;\gamma} h_{\alpha\beta}$$
$$\theta = u^{\alpha}{}_{;\alpha}$$

with $\omega_{(\alpha\beta)} = 0$, $\sigma_{[\alpha\beta]} = 0$ and $\sigma^{\alpha}{}_{\alpha} = 0$ where

$$h_{\alpha\beta} = g_{\alpha\beta} - u_{\alpha} u_{\beta}$$

projects into the 3-rest space of an observer with 4-velocity u^{α}. The observer's acceleration is

$$\dot{u}^{\alpha} = u^{\alpha}{}_{;\beta} u^{\beta} \qquad \text{with} \qquad \dot{u}^{\alpha} u_{\alpha} = 0 \tag{2.1}$$

The Ricci tensor is connected to the scalar quantities $\sigma = \sqrt{(\frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta})}$ and $\omega = \sqrt{(\frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta})}$ by (Ehlers & Kundt, 1962; Ellis, 1971)

$$R_{\alpha\beta} u^{\alpha} u^{\beta} = \dot{\theta} + \tfrac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^{\alpha}{}_{;\alpha} \tag{2.2}$$

3. *Space-like Congruences*

A space-like congruence given by

$$x^{\alpha} = x^{\alpha}(y^{a}, s) \qquad a = 1, 2, 3$$

(where the parameter a labels the particular curve and s is a parameter along the curve) has a normalised tangent vector field k^{α} ($k^{\alpha} k_{\alpha} = -1$). The geometrical properties of the congruence are measured on two-dimensional screens which are placed parallel along and orthogonal to the congruence. The screens are infinitesimally separated and lie in the 3-rest space of an observer u^{α}

$$u^{\alpha} k_{\alpha} = 0 \tag{3.1}$$

The tensor which projects on the 2-spaces of these screens is

$$P_{\alpha\beta} = h_{\alpha\beta} + k_{\alpha} k_{\beta} \tag{3.2}$$

with

$$P_{\alpha\beta} = P_{\beta\alpha}, \qquad P^{\alpha}{}_{\alpha} = 2, \qquad P_{\alpha\beta} k^{\beta} = 0, \qquad P_{\alpha\beta} u^{\beta} = 0, \qquad P_{\alpha\beta} P^{\beta\gamma} = P_{\alpha}{}^{\gamma}$$

We can obtain quantities describing rotation, shear and expansion of the k^{α}-congruence by a method similar to that used for time-like congruences. The vector connecting points with equal parameter s on the neighbouring curves y^{a} and $y^{a} + \delta y^{a}$ is

$$X^{\alpha} = \frac{\partial x^{\alpha}}{\partial y^{a}} \cdot \delta y^{a}.$$

The relative position vector of the two curves as measured on the screen is

$$X^{\alpha}_{\perp} = P^{\alpha}{}_{\beta} X^{\beta} \tag{3.3}$$

and because of (3.1)

$$X^{\alpha}_{\perp} = X^{\alpha} + (X^{\varepsilon} k_{\varepsilon}) k^{\alpha} \tag{3.4}$$

The corresponding relative 'velocity' vector V^{α} representing the change of $X^{\alpha}_{\perp}$ with s as observed on the screen is given by

$$V^{\alpha} = P^{\alpha}{}_{\beta} (X^{\beta}_{\perp})_{;\gamma} k^{\gamma} \tag{3.5}$$

Since second partial derivatives are commutative we have

$$X^{\alpha}{}_{;\beta}k^{\beta} = k^{\alpha}{}_{;\beta}X^{\beta}$$

This enables us to write (3.5) with the use of (3.2) and (3.4) as

$$V_{\alpha} = v_{\alpha\beta}X^{\beta}_{\perp}$$

with

$$v_{\alpha\beta} = P_{\alpha}{}^{\gamma}P_{\beta}{}^{\varepsilon}k_{\gamma;\varepsilon} \tag{3.6}$$

$v_{\alpha\beta}$ describes the linear transformation relating the relative position and the relative 'velocity' of two k^{α}-lines with regard to the 2-screen. We split $v_{\alpha\beta}$ into its antisymmetric part $\omega^{*}_{\alpha\beta} = v_{[\alpha\beta]}$, its trace-free symmetric part $\sigma^{*}_{\alpha\beta} = v_{(\alpha\beta)} - \frac{1}{2}v^{\gamma}{}_{\gamma}P_{\alpha\beta}$ and the trace $\theta^{*} = v^{\alpha}{}_{\alpha}$. Then $\omega^{*}{}_{\alpha\beta}$, $\sigma^{*}{}_{\alpha\beta}$ and θ^{*} are describing respectively the rotation, shear and surface expansion of the k^{α}-congruence on the orthogonal 2-screen in the 3-rest space of the observer u^{α}. According to (3.6) they are given explicitly as

$$\left.\begin{aligned} \omega^{*}{}_{\alpha\beta} &= \tfrac{1}{2}P_{\alpha}{}^{\gamma}P_{\beta}{}^{\varepsilon}(k_{\gamma;\varepsilon} - k_{\varepsilon;\gamma}) \\ \sigma^{*}{}_{\alpha\beta} &= \tfrac{1}{2}P_{\alpha}{}^{\gamma}P_{\beta}{}^{\varepsilon}(k_{\gamma;\varepsilon} + k_{\varepsilon;\gamma}) - \tfrac{1}{2}\theta^{*}P_{\alpha\beta} \\ \theta^{*} &= k^{\alpha}{}_{;\alpha} - u^{\alpha}u^{\beta}k_{\alpha;\beta} \end{aligned}\right\} \tag{3.7}$$

4. *The Gravitational Force*

The gravitational force on a unit mass as measured by an observer with the 4-velocity u^{α} is given by the vector $\dot{u}^{\alpha}$ which lies entirely in the observer's 3-rest space (Trautman, 1964; Dehnen, 1970). This can be shown by discussing the relative 3-acceleration of a freely moving spin-free test particle with regard to an arbitrarily moving observer u^{α}. In other words $\dot{u}^{\alpha}$ is the force which is to be applied on a particle (4-velocity v^{α}) in order to prevent it from falling freely and to keep it at rest ($u^{\alpha} = v^{\alpha}$) relative to a non-geodesically moving observer u^{α}.

The unit vector for the gravitational force ($\dot{u}^{\alpha} \neq 0$) is

$$K^{\alpha} = \frac{\dot{u}^{\alpha}}{K}, \qquad K^{2} = -\dot{u}^{\alpha}\dot{u}_{\alpha}, \qquad K > 0 \tag{4.1}$$

We can describe the geometrical properties of the corresponding force lines by ω^{*}, σ^{*} and θ^{*} of (3.7) in putting $k^{\alpha} = K^{\alpha}$ and using (4.1) which then establishes a connection between the space-like and time-like congruences involved.

5. *Homogeneous Gravitational Force-field*

In order to make the gravitational force-field K^{α} of equation (4.1) appear homogeneous to the observer u^{α}, at least the following *necessary* local conditions must be fulfilled:

(i) The K^{α}-force lines whose 'end points' have the 4-velocity u^{α} are rigid, i.e.,

$$\sigma_{\alpha\beta} = 0, \qquad \theta = 0 \tag{5.1}$$

(ii) The absolute value of the force is constant, i.e.,

$$K_{;\alpha} = 0 \qquad (K \neq 0) \tag{5.2}$$

(iii) The force-field shows neither expansion nor contraction along its lines, i.e.,

$$\theta^* = 0 \tag{5.3}$$

We shall restrict our discussion of (i) to (iii) to a vacuum space-time

$$R_{\alpha\beta} = 0 \tag{5.4}$$

Then (2.2) with (5.1) and (5.4) leads to

$$-2\omega^2 = \dot{u}^{\alpha}{}_{;\alpha} \tag{5.5}$$

and, furthermore, with (4.1) and (5.2) to

$$K^{\alpha}{}_{;\alpha} = -\frac{2\omega^2}{K} \tag{5.6}$$

Because of (2.1) and (4.1) we have

$$u^{\alpha} u^{\beta} K_{\alpha;\beta} = K \tag{5.7}$$

and θ^* of (3.7) therefore takes the form

$$\theta^* = -\frac{1}{K}(2\omega^2 + K^2) \tag{5.8}$$

Equation (5.8) shows that (5.1), (5.2) and (5.4) lead to a non-vanishing and negative θ^* thus indicating a contraction of the force lines in the direction of K^{α}. It is therefore impossible to fulfil in vacuum all three conditions simultaneously which are necessary for the gravitational force-field to be homogeneous.

References

Dehnen, H. (1970). *International Journal of Theoretical Physics*, Vol. 3, No. 6, p. 509.

Ehlers, J. and Kundt, W. (1962). In *Gravitation: an Introduction to Current Research* (Ed. L. Witten). J. Wiley, New York, London.

Ellis, G. F. R. (1971). In *General Relativity and Cosmology* (Ed. R. K. Sachs). Academic Press, New York, London.

Trautman, A. (1964). In *Lectures on General Relativity*, Section 6.6. Prentice Hall, Englewood Cliffs, New Jersey.